Using Graphs to Think About the Jones Polynomial

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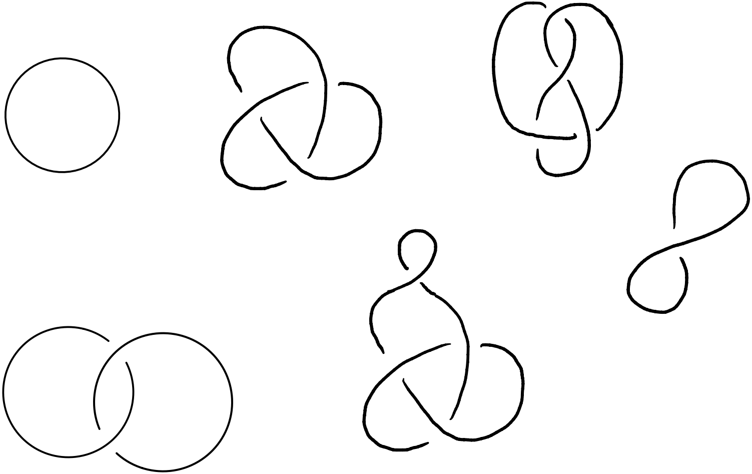
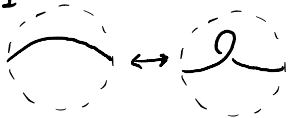
What is a knot?



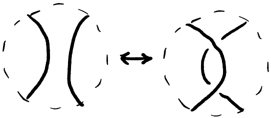
Reidemeister Moves

Reidemeister's Theorem: Let D_1 and D_2 be diagrams of the same knot. Then D_1 and D_2 are related by a sequence of the three Reidemeister moves below.

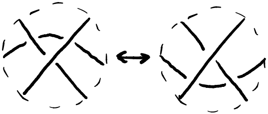
R1



R2



R3



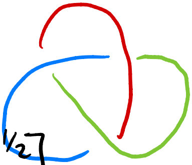
What is a knot invariant?

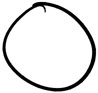
A knot invariant is a property of knots which is the same for equivalent knots. This can be used to distinguish knots.

Examples:

1. Tricolorability
2. Unknotting number
3. Crossing number
4. Jones polynomial

$$\{\text{Knots}\} \to \mathbb{Z}[t^{1/2}, t^{-1/2}]$$
$$\{\text{Knots}\} \to \{\text{yes}, \text{no}\}$$
$$\{\text{Knots}\} \to \mathbb{N}$$





$$2t^{-1/2} + 5t^{2} + 3t^{5/2}$$

Definition: Kauffman Bracket

The Kauffman bracket is characterized by three rules:

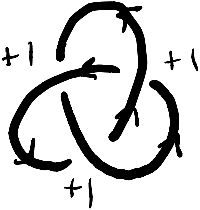
1. $\langle \bigcirc \rangle = 1$
2. $\langle D \sqcup \bigcirc \rangle = (-A^{-2} - A^{2})\langle D \rangle$
3. $\langle$  $\rangle = A\langle$  $\rangle + A^{-1}\langle$  $\rangle$

where D is a link diagram.

Definition: Kauffman Bracket

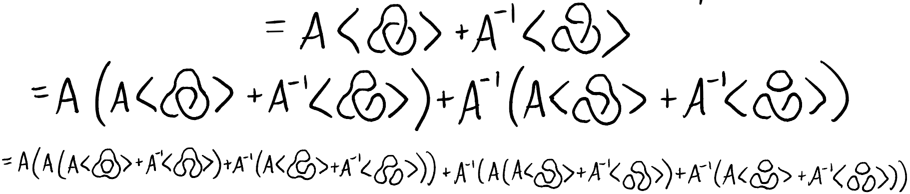
The Kauffman bracket is characterized by three rules:

$\omega(D)=3$



1. $\langle \bigcirc \rangle = 1$
2. $\langle D \sqcup \bigcirc \rangle = (-A^{-2} - A^{2})\langle D \rangle$
3. $\langle \times \rangle = A\langle)(\rangle + A^{-1}\langle \asymp \rangle$

where D is a link diagram.



$= A(A(A(-A^{-2}-A^{2}) + A^{-1}\ \ ) + A^{-1}(\ A\ \ + A^{-1}(-A^{-2}-A^{2}))) + A^{-1}(A(\ A\ \ + A^{-1}(-A^{-2}-A^{2})) + A^{-1}(A(-A^{-2}-A^{2}) + A^{-1}(-A^{-2}-A^{2})^{2}))$

$= A^{3}(-A^{-2}-A^{2}) + 2A + A^{-1}(-A^{-2}-A^{2}) + A + A^{-1}(-A^{-2}-A^{2}) + A^{-1}(-A^{-2}-A^{2}) + A^{-3}(-A^{-2}-A^{2})^{2} = -A^{5} - A^{-3} + A^{-7}$

Definition: Jones Polynomial

Let D be a link diagram of the link L. Then the Jones Polynomial $V(L)$ of L is given by

$$V(L) = \left((-A)^{-3w(D)}\langle D\rangle\right)_{t^{1/2}=A^{-2}} \in \mathbb{Z}\left[t^{-\frac{1}{2}}, t^{\frac{1}{2}}\right]$$

$V($  $)$ $= \left((-A)^{-3\cdot 3}\left(-A^{5}-A^{-3}+A^{-7}\right)\right)_{t^{1/2}=A^{-2}}$

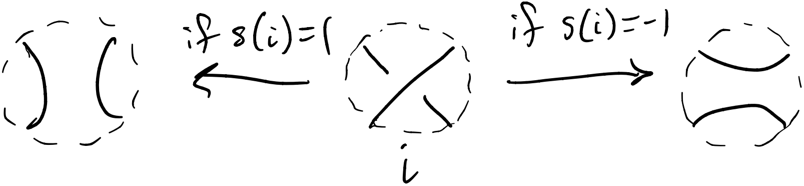
$$= \left(A^{-4}+A^{-12}-A^{-16}\right)_{t^{1/2}=A^{-2}}$$

$$= \left((A^{-2})^{2}+(A^{-2})^{6}-(A^{-2})^{8}\right)_{t^{1/2}=A^{-2}} = \boxed{t+t^{3}-t^{4}}$$

Note: Links with an odd number of components have Jones polynomials with only integer powers of t.

Diagram States

Given a link diagram D with n crossings, a state s of D is a map $s : \{1, 2, ..., n\} \to \{\pm 1\}$,
sD is a diagram which smooths each crossing of D as described below



$|sD|$ is the number of components in sD.

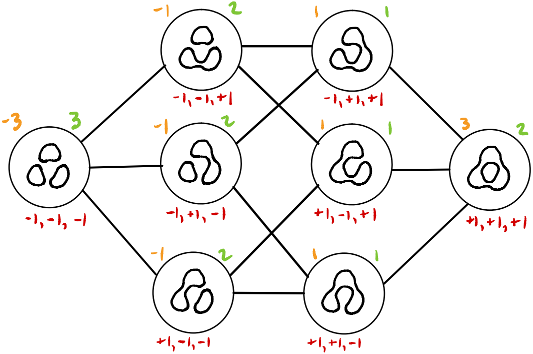
Alternate Definition: Jones Polynomial

Let D be a link diagram of the link L. Then the Jones Polynomial $V(L)$ of L is given by

$$V(L) = \left((-A)^{-3w(D)} \sum_{\substack{\text{states} \\ s \text{ of } D}} \left(A^{\sum_{i=1}^{n} s(i)} (-A^{-2} - A^{2})^{|sD|-1} \right) \right)_{t^{1/2}=A^{-2}}$$

Computing the Jones Polynomial: Trefoil

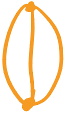
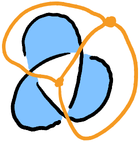
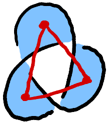
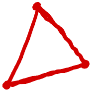
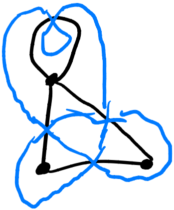
$$V(L) = \left(\left((-A)^{-3w(D)} \sum_S \left(A^{\sum_{i=1}^{n} s(i)} \left(-A^{-2}-A^{2}\right)^{|SD|-1}\right)\right)\right)_{t^{1/2}=A^{-2}} \in \mathbb{Z}[t^{-1/2}, t^{1/2}]$$



$$V(\text{trefoil}) = \Big(\Big((-A)^{-3\cdot 3}\Big(A^{3}(-A^{-2}-A^{2})^{2} + A^{1}(-A^{-2}-A^{2})^{1} + A^{1}(-A^{-2}-A^{2})^{1} + A^{1}(-A^{-2}-A^{2})^{1}$$
$$+ A^{-1}(-A^{-2}-A^{2})^{2} + A^{-1}(-A^{-2}-A^{2})^{2} + A^{-1}(-A^{-2}-A^{2})^{2} + A^{-3}(-A^{-2}-A^{2})^{3}\Big)\Big)\Big)_{t^{1/2}=A^{-2}}$$

Diagram States Using Graphs

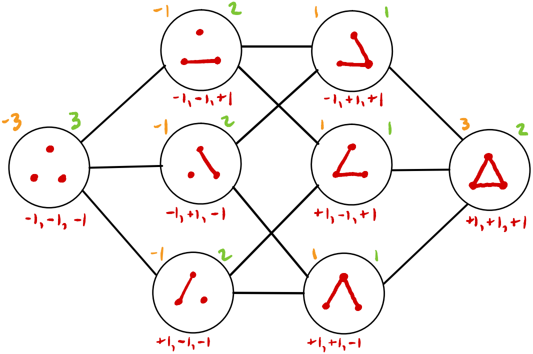
Given any link diagram D, we can checkerboard color the regions of D and construct a graph Γ where the vertices are the black region of D and the edges are the crossings of D between black regions.

Then given a state s of D, we can represent sD by Γ where we delete edges corresponding to smoothings that separate the black regions.


Computing the Jones Polynomial: Trefoil

$$V(L) = \left((-A)^{-3w(D)} \sum_{S} \left(A^{\sum_{i=1}^{n} s(i)} \left(-A^{-2}-A^{2}\right)^{|SD|-1} \right) \right)_{t^{1/2}=A^{-2}} \in \mathbb{Z}[t^{-1/2}, t^{1/2}]$$



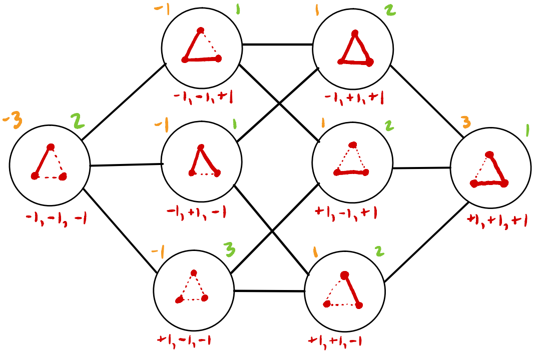
# of components
of regions
-1

$V($$)=$

$$\Big((-A)^{-3\cdot 3} \Big(A^{3}(-A^{-2}-A^{2})^{2} + A^{1}(-A^{-2}-A^{2})^{1} + A^{1}(-A^{-2}-A^{2})^{1} + A^{1}(-A^{-2}-A^{2})^{1}$$
$$+ A^{-1}(-A^{-2}-A^{2})^{2} + A^{-1}(-A^{-2}-A^{2})^{2} + A^{-1}(-A^{-2}-A^{2})^{2} + A^{-3}(-A^{-2}-A^{2})^{3} \Big) \Big)_{t^{1/2}=A^{-2}}$$

Example: Turn That Cube Around

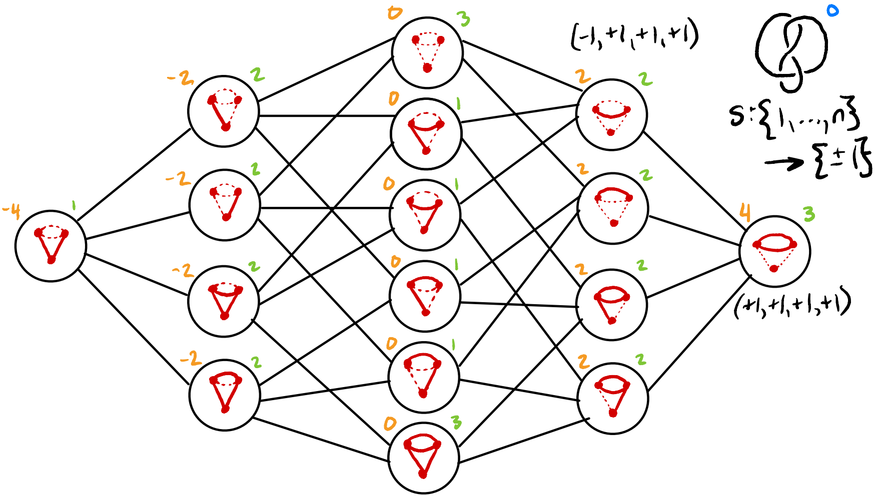
$$V(L) = \left(\left((-A)^{-3w(D)} \sum_S \left(A^{\sum_{i=1}^{n} s(i)} (-A^{-2}-A^{2})^{|SD|-1}\right)\right)\right)_{t^{1/2}=A^{-2}} \in \mathbb{Z}[t^{-1/2}, t^{1/2}]$$



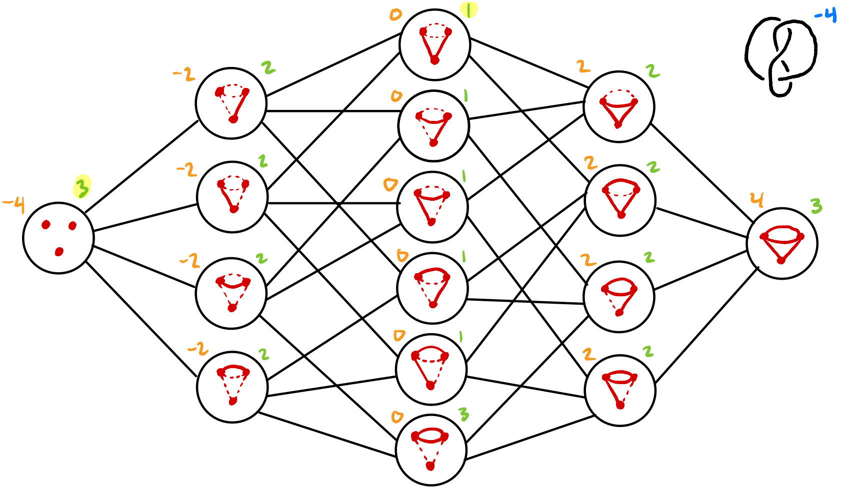
V() =

$$\Big(\Big((-A)^{-3\cdot 1}\big(A^{3}(-A^{-2}-A^{2})^{1}+A^{1}(-A^{-2}-A^{2})^{2}+A^{1}(-A^{-2}-A^{2})^{2}+A^{1}(-A^{-2}-A^{2})^{2}$$
$$+A^{-1}(-A^{-2}-A^{2})^{1}+A^{-1}(-A^{-2}-A^{2})^{1}+A^{-1}(-A^{-2}-A^{2})^{3}+A^{-3}(-A^{-2}-A^{2})^{2}\big)\Big)\Big)_{t^{1/2}=A^{-2}}$$

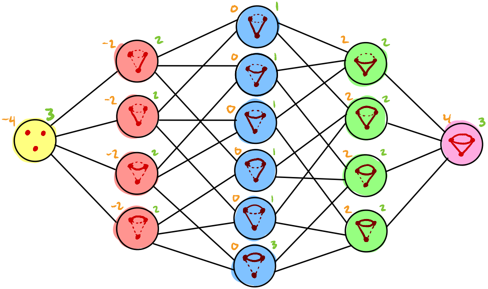
Computing the Jones Polynomial: Figure Eight Knot

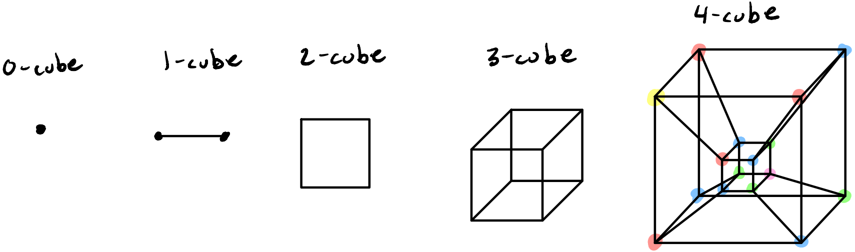


Example: Turn That Hypercube Around



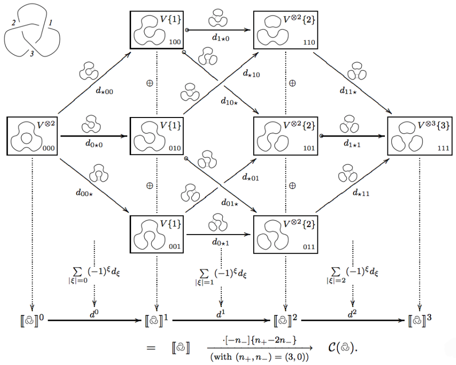
Example: Turn That Hypercube Around





Khovanov Homology

The hypercube of states is useful for defining Khovanov homology, which is a categorification of the Jones polynomial



We Still Don't Understand the Jones Polynomial

Open Questions:

1. Does there exist a nontrivial knot K such that $V(K) = 1$?
2. What is a characterization of Jones polynomials?

References

Dror Bar-Natan
On Khovanov's categorification of the Jones polynomial (2002)

Raymond Lickorish
An Introduction to Knot Theory (1997)